11 Introduction to the Fourier Transform and its Application to PDEs

This is just a brief introduction to the use of the Fourier transform and its inverse to solve some linear PDEs. Actually, the examples we pick just reconfirm d'Alembert's formula for the wave equation, and the heat solution to the Cauchy heat problem, but the examples represent typical computations one must employ to use the technique. The general strategy is to have a PDE for $u(x, t)$ for $x \in \mathbb{R}$ (or for $\mathbf{x} \in \mathbb{R}^n$), use the Fourier transform to get an ODE for the transformed $\hat{u}$ (or a PDE of lower dimensionality if $n > 1$); then solve the ODE and use the inverse Fourier transform (and operational formulas) to get back to a representation for u. Of course, the last step is hardest because it requires algebra and calculus manipulations, and sometimes requiring significant cleverness.

Actually, this is the strategy for all integral transform methods at this level of PDEs. Different types of problems calls for different integral transforms

$$(\mathbb{I}f)(y) := \int_a^b k(y, \zeta) d\zeta = \hat{f}(y) \quad .$$

For example, here are a few of the most common integral transforms:

1. Fourier: $\hat{f}(\xi) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} e^{i\xi x} f(x) dx$
2. Laplace: $F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$
3. Fourier sine: $\hat{f}(\omega) = \mathcal{F}_s[f(x)] = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \sin(\omega x) f(x) dx$
4. Henkel: $F_\nu(k) = \mathcal{H}_\nu[f(r)] = \int_0^{\infty} J_\nu(kr) f(r) r dr$

After our introduction of the Fourier transform, we will briefly review the Laplace transform method in the PDE setting.

11.1 A brief introduction to the Fourier transform

Definition: For any absolutely integrable function $f = f(x)$ defined on $\mathbb{R}$, the Fourier transform of f is given by transform 1 above.

The transform of f in "transform space " can be recovered via an *inversion formula* that defines the **inverse Fourier transform**

$$f(x) = \mathcal{F}^{-1}[\hat{f}(\xi)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} \hat{f}(\xi) d\xi \ \ . \tag{1}$$

Remark: The notation here is not universal. Some authors put the $1/2\pi$ factor in the definition of $\mathcal{F}$, and other authors 'split' it; i.e. put a $1/\sqrt{2\pi}$ factor in the definition of both $\mathcal{F}$ and $\mathcal{F}^{-1}$. It really is incidental but must be kept in mind when looking at Fourier transform tables (or software) from various sources, or working out problems from other books. Although we will formally manipulate the transform as a real integral, the study of integral transforms is best done in an applied complex variables setting.

With every integral transform comes a notion of **convolution**, and here it is defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(x-\xi) g(\xi) d\xi = \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi \ \ . \tag{2}$$

Thus, $f * g = g * f$.

Operational formulas

A. $\mathcal{F}[\frac{d^n f}{dx^n}] = (-i\xi)^n \mathcal{F}[f]$. In particular, $\mathcal{F}[\frac{d^2 f}{dx^2}] = -\xi^2 \hat{f}(\xi)$.

B. shift formula: $\mathcal{F}[f(x-a)] = e^{i\xi a}\mathcal{F}[f(x)]$; so $\mathcal{F}^{-1}[e^{i\xi a}\hat{f}(\xi)] = f(x-a)$.

C. Convolution Theorem: $\mathcal{F}^{-1}[\hat{f}(\xi)\hat{g}(\xi)] = f * g(x)$.

Example: Given $f(x) = H(1-|x|)$, $|x| < \infty$, find $\hat{f}(\xi)$.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} H(1-|x|)\, dx = \int_{-1}^{1} e^{i\xi x}\, dx = \frac{e^{i\xi} - e^{-i\xi}}{i\xi} = \frac{2\sin(\xi)}{\xi} \ .$$

Example: Let $f(x) = s^{-1/2} e^{-x^2/2s^2}$, for $s > 0$. Then

$$\hat{f}(\xi) = s^{-1/2} e^{-s^2\xi^2/2} \int_{-\infty}^{\infty} e^{-(x - is^2\xi)^2/2s^2} dx = \sqrt{2\pi s}\ e^{-(s\xi)^2/2}$$

Example: Solve the ODE $y'' - k^2 y = -f(x)$ on $|x| < \infty$, where we assume $y(x), y'(x) \to 0$ as $|x| \to \infty$. Assume $f(x)$ has a Fourier transform. Taking the Fourier transform of both sides of the equation,

$$-\xi^2 \hat{y} - k^2 \hat{y} = -\hat{f} \quad \to \quad \hat{y} = \frac{\hat{f}}{\xi^2 + k^2} = \hat{f}\ \hat{g}\ .$$

Hence,

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} \frac{d\xi}{\xi^2 + k^2} = \frac{1}{2k} e^{-k|x|}$$

using a transform table (or complex integration theory). So, by the convolution theorem,

$$y(x) = f * g(x) = \frac{1}{2k} \int_{-\infty}^{\infty} e^{-k|x-s|} f(s)\ ds\ .$$

Example: Heat equation on the line

$$\begin{cases} u_t = D u_{xx} & \text{on } |x| < \infty, t > 0 \\ u(x,0) = f(x) & \text{on } |x| < \infty \\ u \text{ remains bounded at infinity} & (|x| \to \infty \text{ and } t \to \infty) \end{cases}$$

Let $\hat{u}(\xi, t) = \int_{-\infty}^{\infty} e^{i\xi x} u(x,t)\ dx$. Then $\hat{u}(\xi, 0) = \int_{-\infty}^{\infty} e^{i\xi x} f(x)\ dx =: \hat{f}(\xi)$. Thus, from property A above, $\mathcal{F}(u_t) = \hat{u}_t = \mathcal{F}(D u_{xx}) = D\mathcal{F}(u_{xx})$, or

$$\begin{cases} \hat{u}_t = -\xi^2 D \hat{u} \\ \hat{u}|_{t=0} = \hat{f}(\xi)\ . \end{cases}$$

This is just a first-order linear equation, so $\hat{u}(\xi, t) = \hat{f}(\xi) e^{-\xi^2 D t}$. Now, if $g(x,t)$ is such that $\hat{g}(\xi, t) = e^{-\xi^2 D t}$, then $\hat{u} = \hat{f}\hat{g}$, and by the convolution theorem,

$$u = f * g, \text{ that is, } u(x,t) = \int_{-\infty}^{\infty} f(\zeta) g(x - \zeta, t) d\zeta\ . \tag{3}$$

So, the task is to find g. By the inversion formula,

$$g(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} e^{-\xi^2 D t} d\xi\ .$$

Note that $-\xi^2 Dt - i\xi x = -Dt(\xi^2 + ix\xi/Dt) = -Dt(\xi + ix/2Dt)^2 - x^2/4Dt$, so

$$2\pi g(x,t) = e^{-x^2/4Dt} \int_{-\infty}^{\infty} e^{-Dt(\xi+ix/2Dt)^2} d\xi.$$

Let $y = \xi + ix/2Dt$, then ($\xi = \pm\infty \to y = \pm\infty$, and $r = y\sqrt{Dt}$)

$$2\pi g(x,t) e^{x^2/4Dt} = \int_{-\infty}^{\infty} e^{-Dty^2} dy = \frac{2}{\sqrt{Dt}} \int_0^{\infty} e^{-r^2} dr = \sqrt{\frac{\pi}{Dt}} \ ,$$

since $\int_0^\infty e^{-r^2} dr = \sqrt{\pi}/2$. Therefore, $g(x,t) = e^{-x^2/4Dt}/(2\sqrt{\pi Dt})$ (which is our previously mentioned fundamental solution to the heat equation), and by (3),

$$u(x,t) = \int_{-\infty}^{\infty} f(\zeta) \frac{e^{-(x-\zeta)^2/4Dt}}{2\sqrt{\pi Dt}} \, d\zeta,$$

which verifies our more informally derived solution to the heat equation done earlier.

Example: 1D wave equation on the line

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{on } |x| < \infty, t > 0 \\ u(x,0) = f(x), u_t(x,0) = g(x) & \text{on } |x| < \infty \\ u \text{ remains bounded at infinity} & (|x| \to \infty \text{ and } t \to \infty) \end{cases}$$

Again let $\mathcal{F}[u] = \hat{u}(\xi,t) = \int_{-\infty}^{\infty} e^{i\xi x} u(x,t) dx$. Then

$$\begin{aligned} &\hat{u}_{tt} = -c^2\xi^2 \hat{u} \\ &\hat{u}(\xi,0) = \hat{f}(\xi), \hat{u}_t(\xi,0) = \hat{g}(\xi) \end{aligned}$$

which implies that $\hat{u}(\xi,t) = A\cos(\xi ct) + B\sin(\xi ct)$; hence, $\hat{u}(\xi,0) = A = \hat{f}(\xi)$, while $\hat{u}_t(\xi,t) = -A\xi c \sin(\xi ct) + B\xi c\cos(\xi ct)$, so that $\hat{u}_t(\xi,0) = B\xi c = \hat{g}(\xi)$. Therefore $\hat{u}(\xi,t) = \hat{f}(\xi)\cos(\xi ct) + (\xi c)^{-1}\hat{g}(\xi)\sin(\xi ct)$. This can not be inverted directly, so replace the trig functions with their exponential representations:

$$\begin{aligned} \hat{u} &= \frac{1}{2}[\hat{f}(\xi) + (\xi ci)^{-1}\hat{g}(\xi)]e^{i\xi ct} + \frac{1}{2}[\hat{f}(\xi) - (\xi ci)^{-1}\hat{g}(\xi)]e^{-i\xi ct} \\ &= \hat{\phi}_+(\xi)e^{i\xi ct} + \hat{\phi}_-(\xi)e^{-i\xi ct}. \end{aligned}$$

By the shift formula, property B above, we have $u(x,t) = \phi_+(x-ct)+\phi_-(x+ct)$, where $a = \mp ct$. We need one other fact, namely

Fact: if $\int_{-\infty}^{\infty} g(x)dx < \infty$, then $\mathcal{F}^{-1}[(i\xi)^{-1}\hat{g}(\xi)] = -\int_{-\infty}^{x} g(y)dy$.

From this we have

$$\phi_\pm(x) = \frac{1}{2}[f(x) \pm c^{-1} \int_{-\infty}^{x} g(y)dy] \ .$$

Combining these results, we have d'Alembert's solution,

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f(x+ct) + c^{-1} \int_{-\infty}^{x+ct} g(y)dy] + \frac{1}{2}[f(x-ct) - c^{-1} \int_{-\infty}^{x-ct} g(y)dy] \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)dy \ , \end{aligned}$$

where we have implicitly assumed $c > 0$.

Remark: In the examples we have used an unstated, but very important property of the Fourier transform (and its inverse). Namely, they are both linear operators, and that fact is used throughout; without the linearity property, the transform would be worthless.

One reason to introduce the Fourier transform now was to reinforce the derived solution expressions for the heat and vibrating string problems on the line by deriving them using the transform method. We'll do a couple more examples here and return to transform methods later.

Example: Laplace's equation on the half space $|x| < \infty, y > 0$
Consider

$$\begin{cases} \nabla^2 u = u_{xx} + u_{yy} = 0 & \text{on } |x| < \infty, y > 0 \\ u(x,0) = f(x) & \text{on } |x| < \infty \\ u \to 0 \text{ at infinity} & \end{cases}$$

Let $\hat{u}(\xi,y) = \int_{-\infty}^{\infty} e^{i\xi x} u(x,y)dx$; upon substituting into the equation gives $\hat{u}_{yy} - \xi^2 \hat{u} = 0$, with $\hat{u}(\xi,0) = \hat{f}(\xi)$, $\hat{u} \to 0$ as $y \to \infty$.
So the solution is a linear combination of $e^{\pm \xi y}$ such that the decay condition holds. Thus, a way of writing $\hat{u}$ is

$$\hat{u}(\xi,y) = Ae^{-|\xi|y}, \text{ or after applying the b.c., } \hat{u}(\xi,y) = \hat{f}(\xi)e^{-|\xi|y}.$$

Thus, $u(x,y) = \mathcal{F}^{-1}[\hat{f}(\xi)e^{-|\xi|y}]$. With $\hat{g} = \hat{g}(\xi, y) = e^{-|\xi|y}$, then if $g(x,y) = \mathcal{F}^{-1}[\hat{g}]$, we have via the convolution theorem, $u = f * g$. Now, from the inversion formula,

$$\begin{aligned} g(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x - |\xi|y} d\xi \\ &= \frac{1}{2\pi}\{\int_{-\infty}^{0} e^{-i\xi x + \xi y} d\xi + \int_{0}^{\infty} e^{-i\xi x - \xi y} d\xi\} \\ &= \frac{1}{2\pi}\{\frac{1}{y - ix} + \frac{1}{y + ix}\} \\ &= \frac{1}{\pi}\frac{y}{x^2 + y^2} . \end{aligned}$$

Finally,

$$u(x,y) = f * g = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2 + y^2}\, ds .$$

Example: Consider

$$\begin{cases} u_t = u_{xx} - ku_x & \text{on } |x| < \infty, t > 0 \\ u(x,0) = f(x) & \text{on } |x| < \infty \end{cases}$$

Hence, $\hat{u}_t = -\xi^2\hat{u} - ik\xi\hat{u}$, $\hat{u}(\xi, 0) = \hat{f}(\xi)$, so $\hat{u}(\xi, t) = \hat{f}(\xi)e^{-\xi^2 t - ik\xi t} = \hat{f}(\xi)\hat{g}(\xi)$. Now

$$g(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\hat{g}(\xi, t)\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t(\xi^2 + i\xi(k + x/t)} d\xi$$

But $\xi^2 + i\xi(\frac{x}{t} + k) = (\xi + \frac{i}{2}(\frac{x}{t} + k))^2 + \frac{1}{4}(\frac{x}{t} + k)^2$. Thus,

$$2\pi g = e^{-t(x/t+k)^2/4}\int_{-\infty}^{\infty} e^{-t(\xi + i(x/t+k)/2)^2} d\xi \quad .$$

Let $r = \sqrt{t}(\xi + \frac{i}{2}(\frac{x}{t} + k))$, then

$$2\pi g = \frac{e^{-t(x/t+k)^2/4}}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-r^2} dr \quad \Rightarrow \quad g(x,t) = \frac{e^{-t(x/t+k)^2/4}}{2\sqrt{\pi t}} .$$

By the convolution theorem,

$$u(x,t) = f * g = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(x-y)e^{-t(k+y/t)^2/4} dy \ .$$

Example:

$$\begin{cases} u_t = t^2 u_{xx} & \text{on } |x| < \infty, t > 0 \\ u(x,0) = f(x) & \text{on } |x| < \infty \end{cases}$$

Now $\hat{u}_t = -t^2\xi^2\hat{u}$, $\hat{u}(\xi,0) = \hat{f}(\xi)$, so $\hat{u}(\xi,t) = \hat{f}(\xi)e^{-\xi^2 t^3/3}$. Let

$$g(x,t) = \mathcal{F}^{-1}[e^{-\xi^2 t^3/3}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}e^{-\xi^2 t^3/3} d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(t^3/3)(\xi^2 + \frac{3ix\xi}{t^3})} d\xi \ .$$

Now $\xi^2 + \frac{3ix}{t^3}\xi = (\xi + \frac{3ix}{2t^3})^2 + \frac{9x^2}{4t^6}$, so

$$2\pi g(x,t) = e^{-3x^2/4t^3}\int_{-\infty}^{\infty} e^{-t^3(\xi + \frac{3ix}{2t^3})^2/3} d\xi \ .$$

Let $r = \frac{t^{3/2}}{\sqrt{3}}(\xi + \frac{3ix}{2t^3})$; then

$$2\pi g(x,t) = \frac{\sqrt{3}e^{-3x^2/4t^3}}{t^{3/2}}\int_{-\infty}^{\infty} e^{-r^2} dr = \frac{\sqrt{3\pi}e^{-3x^2/4t^3}}{t^{3/2}} \quad \text{or} \quad g(x,t) = \frac{1}{2}\sqrt{\frac{3}{\pi t}}\ \frac{e^{-3x^2/4t^3}}{t} \ .$$

Again, by the convolution theorem,

$$u(x,t) = \frac{1}{2}\sqrt{\frac{3}{\pi t^3}}\int_{-\infty}^{\infty} f(\xi)e^{-3(x-\xi)^2/4t^3}\ d\xi \ .$$

Remark: The Fourier transform has a straightforward generalization to higher dimensions. For example, in the plane $\mathbb{R}^2$, given an absolutely integrable f defined on $\mathbb{R}^2$, then

$$\hat{f}(\xi_1,\xi_2) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{i(\xi_1 x + \xi_2 y)} dx\ dy$$

and the inversion formula is

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\xi_1,\xi_2)e^{-i(\xi_1 x + \xi_2 y)} d\xi_1\ d\xi_2$$

Example heat equation in the plane
Consider

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla^2 u & \text{in } \mathbb{R}^2 \times \mathbb{R}^+ \\ u(x,y,0) = f(x,y) & \text{in } \mathbb{R}^2 \\ u \text{ remains bounded at infinity} & \end{cases}$$

First take the Fourier transform in x: $\hat{u}(\xi, y, t) = \int_{-\infty}^{\infty} e^{i\xi x} u(x, y, t) dx$.
Then

$$\begin{cases} \hat{u}_t = -\xi_1^2 \hat{u} + \hat{u}_{yy} \\ \hat{u}(\xi_1, y, 0) = \hat{f}(\xi_1, y) \end{cases}.$$

We now have a 1D diffusion equation in y, t, so now take the Fourier transform in y:

$$\hat{\hat{u}}(\xi_1, \xi_2, t) = \int_{-\infty}^{\infty} e^{i\xi_2 y} \hat{u}(\xi_1, y, t) dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\xi_1 x + \xi_2 y)} u(x, y, t) dx dy.$$

This gives us

$$\begin{cases} \hat{\hat{u}}_t = -(\xi_1^2 + \xi_2^2) \hat{\hat{u}} \\ \hat{\hat{u}}(\xi_1, \xi_2, 0) = \hat{\hat{f}}(\xi_1, \xi_2) \end{cases}$$

Hence,

$$\hat{\hat{u}}(\xi_1, \xi_2, t) = \hat{\hat{f}}(\xi_1, \xi_2) e^{-(\xi_1^2 + \xi_2^2) t}.$$

With the knowledge that $\mathcal{F}^{-1}[e^{-\xi_1^2 t}] = \frac{1}{\sqrt{\pi t}} e^{-x^2/4t}$, we have

$$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi_1 x + \xi_2 y)} e^{-(\xi_1^2 + \xi_2^2) t} d\xi_1 d\xi_2$$

$$= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi_1 x} e^{-\xi_1^2 t} d\xi_1\right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi_2 y} e^{-\xi_2^2 t} d\xi_2\right)$$

$$= \left(\frac{1}{\sqrt{\pi t}} e^{-x^2/4t}\right) \left(\frac{1}{\sqrt{\pi t}} e^{-y^2/4t}\right) = \frac{1}{\pi t} e^{-(x^2 + y^2)/4t}.$$

By the convolution theorem,

$$u(x, y, t) = \frac{1}{\pi t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, s) e^{-[(x-r)^2 + (y-s)^2]/4t} dr ds.$$

Remark: The solution $u(x, y, z, t)$ to the heat equation Cauchy problem in 3D space, with $u(x, y, z, 0) = f(x, y, z)$, is, rather expectantly,

$$u(x, y, z, t) = \frac{1}{(\pi t)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta, \gamma) e^{-\frac{1}{4t}\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}} d\alpha d\beta d\gamma.$$

As mentioned above, we have treated the Fourier transform in a non-rigorous way since it is best handled within the study of complex analysis. But it is valuable for you to get exposure to the technique informally, and see some of its usefulness in getting solutions to some differential equations.

Summary: Know the Fourier and inverse Fourier transform formula and do enough with problems to know the operational formulas.

Exercises

1. Find $\hat{f}(\xi)$, where $f(x) = e^{-ax}\sin(bx)H(x)$, given that a, b are positive constants, and $H(\cdot)$ is the Heaviside function.
   (Answer: $\hat{f}(\xi) = \frac{b}{(a+i\xi)^2+b^2}$)

2. Solve, via the Fourier transform method, the Cauchy problem

   $$u_t = Du_{xx} - Vu_x \quad \text{on } |x| < \infty, t > 0, \text{ with } V, D > 0 \text{ being constants}$$

   (Answer: $u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} f(y)e^{-(x-y-Vt)^2/4Dt}dy$)

3. Revisit the telegraph equation $u_{tt} + 2\beta u_t + \alpha u = c^2 u_{xx}$, $|x| < \infty$, $t > 0$, with $u(x,0) = f(x)$, $u_t(x,0) = g(x)$. Use the Fourier transform to solve the problem in the special case where $\beta^2 = \alpha$.